Inequalities and Asymptotic Formulae Related to Generalizations of the Bessel Functions

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Abstract. We consider some families of 3-index generalizations of the Bessel functions of first kind and study the behaviour of such families in domains of the complex plane. We also prove asymptotic formulae for "large" values of indices of these functions. Similar theorems have also been obtained by the author for the Bessel and Bessel-Maitland functions.

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INTRODUCTION

Let us mention first the so-called Bessel-Clifford functions $C_\nu(z)$, closely related to the Bessel functions $J_\nu(z)$,

$$C_\nu(z) = z^{-\nu/2} J_\nu(2\sqrt{z}) = \sum_{k=0}^{\infty} \frac{(-1)^k (z)^k}{k!\Gamma(k+\nu+1)}, \quad \nu \in \mathbb{C},$$

which are entire functions of z. Generalizations of the Bessel functions (more precisely, of the Bessel-Clifford functions) involving one more additional index μ have been introduced by E.M. (Maitland) Wright [12], called Wright functions or misnamed in the literature also as Bessel-Maitland functions, namely:

$$J_\nu^\mu(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\,\Gamma(\nu+\mu k+1)}, \quad \mu > -1. \tag{1}$$

For details, see Marichev [6], p.109; Kiryakova [3], p.336, [5] etc. Further generalizations of $J_\nu(z)$ have been considered with additional new parameters, like the generalized Bessel-Maitland (or Wright) functions (Pathak [10]):

$$J_{\nu,\lambda}^\mu(z) = (z/2)^{\nu+2\lambda} \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{\Gamma(\lambda+k+1)\Gamma(\nu+k\mu+\lambda+1)}, \; z \in \mathbb{C}\setminus(-\infty,0],\; \mu > 0. \tag{2}$$

In particular ([6], p.110; [3], p. 352-353):

$$J_{\nu,0}^1(z) = J_\nu(z); \quad J_{\nu,0}^\mu(z) = (z/2)^\nu J_\nu^\mu(z^2/4), \quad J_\nu^1(z) = C_\nu(z). \tag{3}$$

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AUXILIARY STATEMENTS

Consider now the generalized Bessel-Maitland (Wright) functions (2) for indices of the kind $\nu = n-2\lambda$; $n = 0,1,2,...,$

$$J^{\mu}_{n-2\lambda,\lambda}(z) = (z/2)^n \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{\Gamma(\lambda+k+1)\Gamma(n-\lambda+k\mu+1)}, \quad z \in \mathbb{C}. \tag{4}$$

Remark 1. Note, that it is possible some coefficients to vanish, depending on the values of the parameter λ, that is there exist numbers $p \in \mathbb{N}_0$, $s \in \mathbb{N}$ such that the identity (4) can be written in the following form

$$J^{\mu}_{n-2\lambda,\lambda}(z) = (z/2)^n \left(\frac{(-1)^p (z/2)^{2p}}{\Gamma(\lambda+p+1)\Gamma(n-\lambda+p\mu+1)} \right. \tag{5}$$

$$\left. + \sum_{k=p+s}^{\infty} \frac{(-1)^k (z/2)^{2k}}{\Gamma(\lambda+k+1)\Gamma(n-\lambda+k\mu+1)} \right).$$

It is not difficult to see that these are entire functions of z ([4]).

Remark 2. We use the denotations $\mathbb{R}^-$ (resp. $\mathbb{R}^+$) for the sets of negative (resp. positive) real numbers, $\mathbb{Z}^-$ (resp. $\mathbb{N}$) for the sets of negative (resp. positive) integers and $\mathbb{Z}^-_0 = \mathbb{Z}^- \cup \{0\}$, and also $a_k = \frac{1}{\Gamma(\lambda+k+1)}$, $b_k = \frac{1}{\Gamma(n-\lambda+k\mu+1)}$, $c_k = a_k b_k$, $k = 0,1,2,\ \dots$.

We consider four main cases.

Lemma 1. *If $\lambda \in \mathbb{C}$, but λ is not negative integer or positive real number, then $p = 0$ and $s = 1$.*

Proof. Obviously, $\lambda+k+1$; $n-\lambda+k\mu+1$ are not non-positive integers and because of that $c_k \neq 0$, for all the values of k. ■

Lemma 2. *If λ is negative integer, then $p = -\lambda$ and $s = 1$.*

Proof. Since λ belongs to $\mathbb{Z}^-$, then $n-\lambda+k\mu+1$ are positive numbers for all k and $\lambda+k+1 \notin \mathbb{Z}^-_0$ for $k \geq -\lambda$, id est $c_k = 0$ for $0 \leq k < -\lambda$ and $c_k \neq 0$ for all $k \geq -\lambda$. ■

Lemma 3. *If λ is positive non-integer real number, then $p = 0$ and $s = 1$ or $s = 2$.*

Proof. In this case a_k are positive numbers for each $k = 0,1,2,...$ and $b_0 \neq 0$ but it possible some of the rest b_k to be zero. Now, let $k = 1$, then two possibilities exist: Or $n-\lambda+\mu+1 \notin \mathbb{Z}^-_0$, and then $b_1 \neq 0$ either $n-\lambda+\mu+1 \in \mathbb{Z}^-_0$, and then $b_1 = 0$. Second case is possible only if $\mu \notin \mathbb{N}$, but then $n-\lambda+2\mu+1 \notin \mathbb{Z}^-_0$ and $b_2 \neq 0$. So, there are two possibilities, namely, $c_0 \neq 0, c_1 \neq 0$ or $c_0 \neq 0, c_1 = 0$, but $c_2 \neq 0$. ■

Lemma 4. *If λ is positive integer, then*
1. $p = 0$ and $s = 1$ for $n \geq \lambda$,
2. $p = 1$ and $s = 1$ or $s = 2$ for $0 \leq n < \lambda$, $\mu \notin \mathbb{N}$,
3. $p = \left[\frac{\lambda-n-1}{\mu}\right] + 1$ and $s = 1$ for $0 \leq n < \lambda$, $\mu \in \mathbb{N}$.

Proof. First, let $n \geq \lambda$. Then $n - \lambda + 1 > 0$, $\lambda + k + 1 > 0$ and therefore all the coefficients $c_k \neq 0$. Now, let $0 \leq n < \lambda$. Then all a_k are positive but $n - \lambda + 1 \in \mathbb{Z}_0^-$ and because of that $b_0 = 0$. Further if $\mu \notin \mathbb{N}$, then $b_1 \neq 0$ and, like in Lemma 3, $b_2 \neq 0$ or $b_2 = 0$ but $b_3 \neq 0$. If $\mu \in \mathbb{N}$, then $n - \lambda + k\mu + 1 \in \mathbb{Z}$ and therefore $b_k = 0$ for $k \leq \frac{\lambda-n-1}{\mu}$, that is $b_k > 0$ for $k > \frac{\lambda-n-1}{\mu}$. So, $c_k = 0$ for $0 \leq k \leq \left[\frac{\lambda-n-1}{\mu}\right]$, and $c_k \neq 0$ for $k \geq \left[\frac{\lambda-n-1}{\mu}\right] + 1$. ■

The above proven lemmas show that the functions $J^{\mu}_{n-2\lambda,\lambda}(z)$ can be written in the form

$$J^{\mu}_{n-2\lambda,\lambda}(z) = \frac{(-1)^p (z/2)^{n+2p}}{\Gamma(\lambda+p+1)\,\Gamma(n-\lambda+p\mu+1)} \left(1 + \theta^{\mu}_{n-2\lambda,\lambda}(z)\right),$$

with

$$\theta^{\mu}_{n-2\lambda,\lambda}(z) = \sum_{k=p+s}^{\infty} (-1)^{k-p} \frac{\Gamma(\lambda+p+1)\Gamma(n-\lambda+p\mu+1)}{\Gamma(\lambda+k+1)\Gamma(n-\lambda+k\mu+1)} \left(\frac{z}{2}\right)^{2(k-p)}, \qquad (6)$$

$$p \geq 0, \quad s \geq 1.$$

MAIN RESULTS

Our aim is to estimate the entire functions $\theta^{\mu}_{n-2\lambda,\lambda}(z)$. To this end we transform the expression in the equality (6) that leads to the identity

$$\theta^{\mu}_{n-2\lambda,\lambda}(z) = \frac{\Gamma(\lambda+p+1)\Gamma(n-\lambda+p\mu+1)}{\Gamma(n-\lambda+(p+s)\mu+1)} \sum_{k=p+s}^{\infty} \frac{(-1)^{k-p}\ \gamma_k}{\Gamma(\lambda+k+1)} \left(\frac{z}{2}\right)^{2(k-p)}, \qquad (7)$$

with

$$\gamma_k = \frac{\Gamma(n-\lambda+(p+s)\mu+1)}{\Gamma(n-\lambda+k\mu+1)}.$$

Remark 3. Remind, for using in the further considerations, that there exists a number $1 < \alpha_0 < 2$ such that Euler's Gamma function $\Gamma(\alpha)$ is on the increase for $\alpha_0 < \alpha < \infty$ and on the decrease for $0 < \alpha < \alpha_0$ and on the set $\mathbb{R}^+$ of the positive real numbers $\Gamma(\alpha)$ has its absolute minimum in the point α_0, that is $\Gamma(\alpha_0) = \min_{\alpha \in \mathbb{R}^+} \Gamma(\alpha)$.

Theorem 1. *Let* $\lambda = 0$. *Then*

$$|\theta^{\mu}_{n-2\lambda,\lambda}(z)| \leq \frac{\Gamma(n+1)}{\Gamma(n+\mu+1)} \left(\exp\left(\left|\frac{z^2}{4}\right|\right) - 1\right), \quad n \in \mathbb{N},$$

$$|\theta^{\mu}_{n-2\lambda,\lambda}(z)| \leq \frac{1}{\Gamma(\mu+1)\Gamma(\alpha_0)} \left(\exp\left(\left|\frac{z^2}{4}\right|\right) - 1\right), \quad n = 0.$$

Proof. Because of Lemma 1, we have $p=0,\ s=1,\ \ \gamma_k=\frac{\Gamma(n+\mu+1)}{\Gamma(n+k\mu+1)}$ and

$$\theta^{\mu}_{n-2\lambda,\lambda}(z)=\frac{\Gamma(n+1)}{\Gamma(n+\mu+1)}\sum_{k=1}^{\infty}\frac{(-1)^k\ \gamma_k}{\Gamma(k+1)}\ \left(\frac{z}{2}\right)^{2k},$$

whence

$$|\theta^{\mu}_{n-2\lambda,\lambda}(z)|\le\frac{\Gamma(n+1)}{\Gamma(n+\mu+1)}\sum_{k=1}^{\infty}\frac{\left(|z/2|^2\right)^k}{\Gamma(k+1)}\,\gamma_k.$$

In view of Remark 3, we can write $0<\gamma_k\le 1$ for $n\in\mathbb{N}$ and $0<\gamma_k\le\frac{\Gamma(\mu+1)}{\Gamma(\alpha_0)}$ for $n=0$ that proves the theorem. ■

Theorem 2. *Let $\lambda\in\mathbb{Z}^-$, then*

$$|\theta^{\mu}_{n-2\lambda,\lambda}(z)|\le\Gamma(1+\mu-\lambda\mu)\,\frac{\Gamma(n-\lambda-\lambda\mu+1)}{\Gamma(n-\lambda+(1-\lambda)\mu+1)}\left(J^{\mu}_{-\lambda\mu}\left(-\left|\frac{z^2}{4}\right|\right)-\frac{1}{\Gamma(1-\lambda\mu)}\right).$$

Proof. According to Lemma 2, $p=-\lambda$ and $s=1$ and therefore (7) takes the form

$$\theta^{\mu}_{n-2\lambda,\lambda}(z)=\frac{\Gamma(n-\lambda-\lambda\mu+1)}{\Gamma(n-\lambda+(1-\lambda)\mu+1)}\sum_{k=p+1}^{\infty}\frac{(-1)^{k-p}\ \gamma_k}{\Gamma(\lambda+k+1)}\ \left(\frac{z}{2}\right)^{2(k-p)},$$

with

$$\gamma_k=\frac{\Gamma(n-\lambda+(p+1)\mu+1)}{\Gamma(n-\lambda+k\mu+1)}.$$

Since

$$\gamma_k=\frac{(n-\lambda+(p+1)\mu)}{(n-\lambda+k\mu)}\times\ldots\times\frac{(1+(p+1)\mu)}{(1+k\mu)}\times\frac{\Gamma(1+(p+1)\mu)}{\Gamma(1+k\mu)},$$

then

$$0<\gamma_k\le\frac{\Gamma(1+(1-\lambda)\mu)}{\Gamma(1+k\mu)}.$$

Using denotation

$$w_k(z)=\frac{(-1)^{k-p}\ \gamma_k}{\Gamma(\lambda+k+1)}\ \left(\frac{z}{2}\right)^{2(k-p)},$$

we obtain consecutively

$$\sum_{k=p+1}^{\infty}w_k(z)=\sum_{k=1}^{\infty}w_{k+p}(z)=\sum_{k=1}^{\infty}\frac{(-1)^k\ \gamma_{k+p}}{\Gamma(k+1)}\ \left(\frac{z}{2}\right)^{2k},$$

$$\left|\sum_{k=p+1}^{\infty}w_k(z)\right|\le\sum_{k=1}^{\infty}\frac{\gamma_{k+p}}{\Gamma(k+1)}\ \left(\left|\frac{z}{2}\right|^2\right)^k\le\sum_{k=1}^{\infty}\frac{\Gamma(1+\mu-\lambda\mu)\left(|z/2|^2\right)^k}{\Gamma(k+1)\Gamma(1+k\mu-\lambda\mu)}$$

$$= \Gamma(1+\mu-\lambda\mu)\left(\sum_{k=0}^{\infty}\frac{\left(|z/2|^2\right)^k}{\Gamma(k+1)\Gamma(1+k\mu-\lambda\mu)}-\frac{1}{\Gamma(1-\lambda\mu)}\right),$$

whence the conclusion of the theorem immediately follows. ■

Theorem 3. *Let $\lambda \in \mathbb{R}^- \setminus \mathbb{Z}$, then there exists an entire function τ such that*

$$|\theta^{\mu}_{n-2\lambda,\lambda}(z)| \leq |\Gamma(\lambda+1)|\,\frac{\Gamma(n-\lambda+1)}{\Gamma(n-\lambda+\mu+1)}\,\sum_{k=1}^{\infty}\frac{(|z/2|)^{2k}}{|\Gamma(\lambda+k+1)|}\tau(|z|^2/4;\lambda).$$

Proof. Now, due to Lemma 1, we have $p=0,\ s=1,\ \ \gamma_k = \frac{\Gamma(n-\lambda+\mu+1)}{\Gamma(n-\lambda+k\mu+1)}$ and

$$\theta^{\mu}_{n-2\lambda,\lambda}(z) = \Gamma(\lambda+1)\frac{\Gamma(n-\lambda+1)}{\Gamma(n-\lambda+\mu+1)}\,\sum_{k=1}^{\infty}\frac{(-1)^k\ \gamma_k}{\Gamma(\lambda+k+1)}\,\left(\frac{z}{2}\right)^{2k}, \tag{8}$$

respectively

$$|\theta^{\mu}_{n-2\lambda,\lambda}(z)| \leq |\Gamma(\lambda+1)|\frac{\Gamma(n-\lambda+1)}{\Gamma(n-\lambda+\mu+1)}\,\sum_{k=1}^{\infty}\frac{\left(|z/2|^2\right)^k}{|\Gamma(\lambda+k+1)|}\,\gamma_k.$$

Further, the increase of the Gamma function leads to the inequality $0<\gamma_k\leq 1$ for $n\in\mathbb{N}$ and $n=0$ but $1-\lambda+\mu\geq\alpha_0$ whereas $0<\gamma_k\leq\frac{1}{\Gamma(\alpha_0)}$ for $n=0$ and $1<1-\lambda+\mu<\alpha_0$. Finally, getting $C=\max\left(1,\frac{1}{\Gamma(\alpha_0)}\right)=\frac{1}{\Gamma(\alpha_0)}$ and $\tau(z;\lambda)=C\sum_{k=1}^{\infty}\frac{z^k}{|\Gamma(\lambda+k+1)|}$, we complete the proof of the theorem. ■

Theorem 4. *Let $\lambda\notin\mathbb{R}$, that is $\lambda=\lambda_1+i\lambda_2$ ($\lambda_1,\lambda_2\in\mathbb{R},\ \lambda_2\neq 0$), then there exists entire functions φ and φ_1 such that*

$$|\theta^{\mu}_{n-2\lambda,\lambda}(z)| \leq |\Gamma(\lambda+1)|\,\frac{|\Gamma(n-\lambda+1)\,|}{|\Gamma(n-\lambda+\mu+1)|}\,\varphi(|z|^2/4;\lambda,\mu)$$

for $n\geq\lambda_1$ and

$$|\theta^{\mu}_{n-2\lambda,\lambda}(z)| \leq |\Gamma(\lambda+1)|\,\frac{|\Gamma(n-\lambda+1)\,|}{|\Gamma(n-\lambda+\mu+1)|}\,\varphi_1(|z|^2/4;\lambda,\mu)$$

for $0\leq n<\lambda$.

Proof. Analogously to Theorem 3, $p=0,\ s=1$, and identity (8), like in Theorem 3 and with the same γ_k, expresses $\theta^{\mu}_{n-2\lambda,\lambda}(z)$.

First, we consider the case $n\geq\lambda_1$. Following the idea of the proof of Theorem 2, we obtain the estimate

$$|\gamma_k| \leq \frac{|\Gamma([\lambda_1]+1-\lambda_1+\mu-i\lambda_2)|}{|\Gamma([\lambda_1]+1-\lambda_1+k\mu-i\lambda_2)|} = \frac{|\Gamma([\lambda_1]+1-\lambda+\mu)|}{|\Gamma([\lambda_1]+1-\lambda+k\mu)|}.$$

Now, let $0 \leq n < \lambda_1$. Then because of the convergency of the sequence $\{\gamma_k\}_{k=1}^{\infty}$, it is bounded and therefore there exists a constant $\tilde{C}$ such that $|\gamma_k| \leq \tilde{C}$ for all the values of k.

Eventually, the proof ends taking

$$\varphi(z;\lambda,\mu) = |\Gamma([\lambda_1]+1-\lambda+\mu)| \sum_{k=1}^{\infty} \frac{z^k}{|\Gamma(\lambda+k+1)\Gamma([\lambda_1]+1-\lambda+k\mu)|},$$

and $\varphi_1(z;\lambda,\mu) = \tilde{C} \sum_{k=1}^{\infty} \frac{z^k}{|\Gamma(\lambda+k+1)|}$. ∎

Let $E_{\alpha,\beta}(z)$ and $E_{(1/\rho_1,1/\rho_2),(\mu_1,\mu_2)}$ be respectively the Mittag-Leffler function and its multi-index (two-index) analogue, namely

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \qquad (\alpha > 0,\ \beta > 0),$$

$$E_{(1/\rho_1,1/\rho_2),(\mu_1,\mu_2)}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma\left(\mu_1+\frac{k}{\rho_1}\right)\Gamma\left(\mu_2+\frac{k}{\rho_2}\right)}, \quad (\rho_1 > 0,\ \rho_2 > 0).$$

The detailed properties of these functions can be found in the contemporary monographs [2], [3], and [11], for details see [4], [5].

Theorem 5. *Let $\lambda \in \mathbb{R}^+ \setminus \mathbb{N}$, then for $n > \lambda$*

$$|\theta^{\mu}_{n-2\lambda,\lambda}(z)| \leq \Gamma(\lambda+1)\Gamma([\lambda]-\lambda+s\mu+1)\frac{\Gamma(n-\lambda+1)}{\Gamma(n-\lambda+s\mu+1)}$$
$$\times \left(E^{(2)}_{(1,\mu),(\lambda+1,[\lambda]-\lambda+1)}\left(|z|^2/4\right) - \frac{1}{\Gamma(\lambda+1)\Gamma([\lambda]+1-\lambda)}\right),$$

and there exists a constant C such that for $0 \leq n < \lambda$

$$|\theta^{\mu}_{n-2\lambda,\lambda}(z)| \leq C\Gamma(\lambda+1)\frac{|\Gamma(n-\lambda+1)|}{|\Gamma(n-\lambda+s\mu+1)|}\left(E_{(1,\lambda+1)}\left(|z|^2/4\right) - \frac{1}{\Gamma(\lambda+1)}\right).$$

Proof. According to Lemma 3, $p = 0,\ s = 1$ or $s = 2$, $\gamma_k = \frac{\Gamma(n-\lambda+s\mu+1)}{\Gamma(n-\lambda+k\mu+1)}$ and

$$\theta^{\mu}_{n-2\lambda,\lambda}(z) = \Gamma(\lambda+1)\frac{\Gamma(n-\lambda+1)}{\Gamma(n-\lambda+s\mu+1)} \sum_{k=s}^{\infty} \frac{(-1)^k\ \gamma_k}{\Gamma(\lambda+k+1)} \left(\frac{z}{2}\right)^{2k},$$

respectively

$$|\theta^{\mu}_{n-2\lambda,\lambda}(z)| \leq \Gamma(\lambda+1)\left|\frac{\Gamma(n-\lambda+1)}{\Gamma(n-\lambda+s\mu+1)}\right| \sum_{k=s}^{\infty} \frac{\left(|z/2|^2\right)^k}{\Gamma(\lambda+k+1)}\,|\gamma_k|.$$

For $n > \lambda$, in the same way like in the proof of Theorem 4 we get to the inequality

$$0 < \gamma_k \leq \frac{\Gamma([\lambda]+1-\lambda+s\mu)}{\Gamma([\lambda]+1-\lambda+k\mu)}$$

for all $k \geq s$ that immediately ends the proof.

The case $n < \lambda$, again goes in the same way like in Theorem 4. ■

Theorem 6. *Let $\lambda \in \mathbb{N}$, then*

$$|\theta^{\mu}_{n-2\lambda,\lambda}(z)| \leq \Gamma(\lambda+1)\,\Gamma(\mu+1)\,\frac{\Gamma(n-\lambda+1)}{\Gamma(n-\lambda+\mu+1)}\left(E^{(2)}_{(\mu,1)(1,\lambda+1)}\left(|z|^2/4\right) - \frac{1}{\Gamma(\lambda+1)}\right)$$

for all $n \geq \lambda$ and there exists a constant C such that for $0 \leq n < \lambda$ holds

$$|\theta^{\mu}_{n-2\lambda,\lambda}(z)| \leq C\,\Gamma(\lambda+p+1)\,\frac{\Gamma(n-\lambda+p\mu+1)}{\Gamma(n-\lambda+(p+s)\mu+1)}\left(E_{(1,\lambda+1)}\left(|z|^2/4\right) - \frac{1}{\Gamma(\lambda+1)}\right)$$

with the corresponding p and s.

Proof. Beginning with the case $n \geq \lambda$, we have, according to Lemma 4, that $p = 0$, $s = 1$ and identity (8), like in the proof of Theorem 3, expresses $\theta^{\mu}_{n-2\lambda,\lambda}(z)$, with the same γ_k. Then, like in the proof of Theorem 2, we get to the inequalities $0 < \gamma_k \leq \frac{\Gamma(\mu+1)}{\Gamma(k\mu+1)}$ for all $k \in \mathbb{N}$ and therefore

$$|\theta^{\mu}_{n-2\lambda,\lambda}(z)| \leq \Gamma(\lambda+1)\frac{\Gamma(n-\lambda+1)}{\Gamma(n-\lambda+\mu+1)}\sum_{k=1}^{\infty}\frac{\left(|z/2|^2\right)^k}{\Gamma(\lambda+k+1)}\,\gamma_k$$

$$\leq \frac{\Gamma(n-\lambda+1)\,\Gamma(\lambda+1)\,\Gamma(\mu+1)}{\Gamma(n-\lambda+\mu+1)}\left(\sum_{k=0}^{\infty}\frac{\left(|z/2|^2\right)^k}{\Gamma(k\mu+1)\Gamma(\lambda+k+1)} - \frac{1}{\Gamma(\lambda+1)}\right)$$

$$= \frac{\Gamma(n-\lambda+1)\,\Gamma(\lambda+1)\,\Gamma(\mu+1)}{\Gamma(n-\lambda+\mu+1)}\left(E^{(2)}_{(\mu,1)(1,\lambda+1)}\left(|z|^2/4\right) - \frac{1}{\Gamma(\lambda+1)}\right)$$

for all $n \geq \lambda$.

The proof of the case $n < \lambda$ become in the same way like in Theorem 4, using the inequalities $\Gamma(\lambda+k+p+1) > \Gamma(\lambda+k+1) > 0$. ■

CONCLUSION

By the way, using connection (3), similar estimates for functions like (1) and the Bessel functions can be obtained, just as a particular case of the above given. Such inequalities are recently considered and used by the author to study series convergency ([7]-[9]). By means of these estimates and using Stirling's formula (see e.g. [1]) one can obtain the following result.

Theorem 7. *Let $\mu > 0$. Then the generalized Bessel-Maitland (Wright) functions (4) satisfy the following asymptotic formula*

$$J^{\mu}_{n-2\lambda,\lambda}(z) = \frac{(-1)^p (z/2)^{n+2p}}{\Gamma(\lambda+p+1)\Gamma(n-\lambda+p\mu+1)}(1+\theta^{\mu}_{n-2\lambda,\lambda}(z)), \tag{9}$$

$$\theta^{\mu}_{n-2\lambda,\lambda}(z) \to 0 \;\; as \;\; n\to\infty.$$

On the compact subsets of the complex plane $\mathbb{C}$ convergence is uniform and

$$\theta^{\mu}_{n-2\lambda,\lambda}(z) \; = \; O\left(\frac{1}{n^{s\mu}}\right), \quad (n \;\in \mathbb{N}) \tag{10}$$

with the corresponding $s \in \mathbb{N}$, depending on λ.

The proof is evident. ∎

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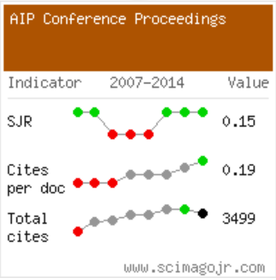

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